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A General Strength Distribution Function for Brittle Materials

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Abstract

A new strength distribution function for brittle materials is developed, which applies to materials with an inhomogeneous distribution of flaws.

The probability of failure is

$$F = 1 - \exp\left[-\langle N_{c,s}\rangle\right]$$

where $\langle N_{c,s}\rangle$ is the mean number of critical defects in the specimen of size S. The well-known Weibull statistics are a special case of the new statistics for a special flaw size distribution.

Several aspects of the relationships between the Weibull statistics and material structure are analysed in the light of the new formalism. Examples are materials with several different flaw distributions or rising crack resistance. The conditions necessary to get a Weibull distribution as well as the reasons why Weibull distributions are observed so often in the daily material testing practice are discussed. Finally, the minimum number of test specimens necessary to guarantee a reliable prediction of the component's reliability using Weibull's theory is given. This number depends on the necessary reliability as well as on the loaded (effective) volumes of the test specimens and components, respectively.

Es wurde eine neue Festigkeitsverteilungsfunktion für spröde Werkstoffe entwickelt, die Werkstoffe mit einer inhomogenen Fehlerverteilung beschreibt.

Die Bruchwahrscheinlichkeit ergibt sich aus

$$F = 1 - \exp\left[-\langle N_{c,s}\rangle\right]$$

Hierbei ist $\langle N_{c,s}\rangle$ die mittlere Anzahl kritischer Defekte in einer Probe der Größe S. Die bekannte Weibullstatistik ist ein Sonderfall der neuen Statistik und zwar für eine besondere Fehlergrößenverteilung.

Verschiedene Aspekte der Beziehungen zwischen der Weibullstatistik und der Materialstruktur werden aus der Sicht des neuen Formalismus analysiert. Beispiele hierzu bilden Werkstoffe mit mehreren verschiedenen Fehlerverteilungen oder mit zunehmendem Rißwiderstand. Die erforderlichen Bedingungen, die zu einer Weibull-Verteilung führen, werden diskutiert. Außerdem werden die Ursachen, weshalb die Weibull-Verteilung so häufig in der täglichen Praxis der Werkstoffprüfung beobachtet wird, besprochen. Schließlich wird die minimale Probenanzahl angegeben, die für eine Gewährleistung einer sicheren Vorhersage der Zuverlässigkeit der Komponente nach der Weibull-Theorie erforderlich ist. Diese Anzahl hängt sowohl von der erforderlichen Zuverlässigkeit als auch von dem effektiv belasteten (effektiven) Volumen der Probe bzw. der Komponente ab.

Une nouvelle fonction de distribution de la résistance mécanique pour des matériaux cassant a été développée, qui s'applique aux matériaux avec une distribution inhomogène des défauts.

La probabilité de défauts est

$$F = 1 - \exp\left[-\langle N_{c,s}\rangle\right]$$

où $\langle N_{c,s}\rangle$ est le nombre de défauts critiques moyen dans l'échantillon de taille S. La statistique bien connue de Weibull est un cas particulier de la nouvelle statistique pour une distribution particulière de défauts.

Plusiers aspects de la relation entre la statistique Weibull et la structure du matériau sont analysés à la lumière de ce nouveau formalisme. Des exemples sont des matériaux avec plusieurs distributions de défauts ou une résistance croissante à la fissure. Les conditions nécessaires pour avoir une distribution Weibull, aussi bien que les raisons pour lesquelles des distributions Weibull sont observées si souvent dans la pratique *courante des tests sur les matériaux, sont discutées. Finalement on a donné le nombre minimum d'échantillons d'essais nécessaires pour garantir une bonne mesure de la fiabilité du composant en utilisant la théorie de Weibull. Ce nombre dépend de la fiabilité nécessaire ainsi que des volumes en charge (effective) des échantillons d'essais et des composants.*

1 Introduction

In brittle materials, e.g. ceramics, fracture usually starts from defects. The material strength then depends on the strength of the major defect in the material, which varies from specimen to specimen. Therefore the strength of brittle materials is not given by a simple number. For a set of identical specimens, a strength distribution function is required which describes the probability of failure depending on the stress state and amplitude. The design of components made from brittle materials is based on the knowledge of this distribution function, which can only be measured on a large set of test specimens. This is expensive and will hardly be done in the daily design practice.

The experimental efforts necessary to find the proper strength distribution can be reduced to a large extent if its mathematical structure can be deduced from physical principles. On the basis of a known functional context between the stress state and the probability of failure, some material parameters instead of the whole distribution curve have to be measured. In the past, several distribution functions of strength were derived using probabilistic arguments.[1–10] It has always been assumed in these papers that the defects are homogeneously distributed within the specimen. This is hardly the case in real materials. In this paper, a more general distribution function of strength for brittle materials with an inhomogeneous density of defects is deduced. It can be shown that the other distribution functions, e.g. the well-known Weibull statistics,[1,2] are special cases of this new function. The new fracture statistics are used to describe several typical situations occurring in specimens made of brittle materials.

2 Fracture Statistics of Brittle Materials

Brittle fracture designates a group of fracture processes which are neither preceded nor accompanied by high degree of plastic deformation. Fracture in brittle materials originates at defects. These are regions where stresses are concentrated by the microstructure, e.g. defects can be crack-like flaws or flaws which, under the action of a stress, can transform into them. A defect is called critical if the tensile stresses in these regions are high enough to destroy the cohesion of the material. The minimum size of the critical defect depends on the local material's structure, the kind of defect (and the corresponding local fracture criterion), on the shape, orientation of the defect, and on the macroscopic stress state at the site of the defect. Since only one defect can actually be the nucleation site of brittle fracture in any given sample, there is a need for more complex statistics which yield both sample size and stress dependencies, which is the main point of this paper. (At a high stress amplitude more than one defect may be weaker than the applied stress; those defects are *potentially critical.* In an actual loading situation, however, the load is increased up to the analysed amplitude. Fracture occurs when the tensile stress amplitude at the site of the weakest defect exceeds the strength of the weakest defect. This is the *actually critical* defect.)

In general, the size of a critical defect correlates with the amplitude of the stress state (the higher the defect size, the lower the stress amplitude (strength)). The microscopic fracture criterion (as well as the proper definition of the size of a critical defect depending on the stress amplitude) may be different for different kinds of flaws. This problem will not be tackled in this paper. It is assumed that such a definition exists (an example will be given later) and that, for a given stress state and in a given volume element, a density of critical defects can be defined. In reliable components, this density is low and it has to be measured in a large set of specimens (components). The density of (potentially and actually) critical defects is equal to the mean number (taken over this set of specimens) of critical defects divided by the size of the volume element. The density of critical defects can vary within the specimen. This is reflected by the fact that critical areas exist in most components. Reasons for such an inhomogeneity are an inhomogeneous stress state, an inhomogeneous microstructure or an inhomogeneous defect distribution.

In the following a general fracture statistical relationship between the density of critical defects and the reliability of a specimen is derived. In a later section, it is demonstrated on several examples how this relationship can be used to obtain the fracture statistics (strength distribution) of a set of specimens.

In order to derive the probability of failure, it is assumed that:

(a) The density of defects (defined in a volume element which is large enough to contain several defects) is low enough so that interaction between flaws can be neglected;
(b) a brittle material fails when the weakest defect fails, like a chain breaking when the weakest link fails (weakest link hypothesis); and
(c) a density of critical defects, ρ_c, can be defined for a set of microscopically identical specimens.

Under the action of a given stress state, a specimen will fail if it contains at least one critical defect (according to assumption (b)) and it will not fail if it does not contain any critical defect. Therefore, the reliability (probability of not failing), R, is equal to the probability of finding no critical defect within the specimen. For obvious reasons, the probability of failure, F, is given by

$$F = 1 - R$$

Let us consider the probability of finding a critical volume defect (the probability of failure) in a small volume element dV'. It is proportional to this volume and to the local density of critical volume defects, $\rho_{c,v}(V')$ (which exists according to assumption (c)). In a specimen with volume V, this probability is a continuous and monotonically increasing function of V according to the integral

$$\int_V \rho_{c,v}(V')\,dV' = \langle N_{c,v}(V) \rangle \qquad (1)$$

and F is the mean number (taken over a large set of microscopically identical specimens) of critical volume defects in the volume V:

$$F = F[\langle N_{c,v}(V) \rangle]$$

Because $R = 1 - F$, the reliability is a continuous and monotonically decreasing function of $\langle N_{c,v}(V) \rangle$ and of V, therefore

$$R = R[\langle N_{c,v}(V) \rangle]$$

This expression is abbreviated by the symbol $R(V)$. The later derivation is guided by the ideas which Freudenthal[4] has developed for materials with a homogeneous defect density.

Let $R(V + V_1)$ be the probability of finding no critical volume defect within the volume $V + V_1$. According to assumption (a), the reliability of finding no critical volume defect in V does not depend on the reliability of finding no critical volume defect in V_1 and it holds that

$$R(V + V_1) = R(V)R(V_1) \qquad (2)$$

The theorem of Leibniz–Newton is used to calculate the total differential of the mean number of critical volume defects in V (eqn (1)):

$$d\langle N_{c,v}(V) \rangle = d\left[\int_V \rho_{c,v}(V')\,dV'\right] = \rho_{c,v}(V)\,dV \qquad (3)$$

The change of reliability with a change of the mean number of critical volume defects in V is

$$\frac{dR(V+V_1)}{d\langle N_{c,v}(V) \rangle} = \frac{dR(V+V_1)}{\rho_{c,v}(V)\,dV} = R(V_1)\frac{dR(V)}{\rho_{c,v}(V)\,dV} \qquad (4)$$

Dividing by eqn (2) gives

$$\frac{d\ln R(V+V_1)}{\rho_{c,v}(V)\,dV} = \frac{d\ln R(V)}{\rho_{c,v}(V)\,dV} \qquad (5)$$

The left-hand side of eqn (5) depends on the mean number of critical volume defects in V_1, but the right-hand side does not. Equation (5) is valid for all values of V_1, and therefore both sides have to be a constant value (called c_0) not depending on the mean number of critical volume defects in the volume V_1. Now eqn (5) can easily be integrated by separation of the variables

$$d\ln R(V) = c_0 \rho_{c,v}(V)\,dV \qquad (6)$$

and

$$\ln R(V) = c_0 \langle N_{c,v}(V) \rangle + c_1 \qquad (7)$$

where eqn (1) has been used and c_1 is an integration constant. For $V \to 0$, $\langle N_{c,v}(V) \rangle \to 0$ and $R(0)$ has to be unity, giving $c_1 = 0$ and

$$R(V) = \exp[c_0 \langle N_{c,v}(V) \rangle]$$

For small volume elements, the probability of failure has to be equal to the probability of finding a critical defect in the volume. Therefore, $c_0 = -1$ and

$$R(V) = \exp[-\langle N_{c,v}(V) \rangle] \qquad (8)$$

This is the reliability function for brittle specimens containing volume defects. No special assumptions have been made concerning the loading, the kind of defects and the fracture criterion; their influence is incorporated in the definition of the 'critical' volume defects. It is possible that the specimens contain several populations of volume defects which may fail due to different reasons, and that the density of critical volume defects is inhomogeneous.

An analogous calculation can be done for surface defects (density: $\rho_{c,a}(A)$; area: A; mean number of critical surface defects in a specimen: $\langle N_{c,a}(A) \rangle$) and for edge defects (density: $\rho_{c,l}(L)$; edge length: L; mean number of critical edge defects in a specimen: $\langle N_{c,l}(L) \rangle$). Both kinds of defects are frequently produced during machining of specimens. If the probabilities of failure due to different defect populations are independent, the total reliability of the specimen S is the product of the reliabilities, R_i, corresponding to the individual defect populations, i:

$$R_S = \prod_i R_i = \exp\left[-\sum_i \langle N_{c,i}\rangle\right] \quad (9)$$

and

$$R_S = \exp[-\langle N_{c,S}\rangle] \quad (10)$$

respectively

$$F_S = 1 - \exp[-\langle N_{c,S}\rangle] \quad (11)$$

The subscript S refers to size and geometry of the specimen, $\langle N_{c,S}\rangle$ to the mean value (taken over a set of microscopically identical specimens loaded in an identical way) of finding any critical defect in a specimen (volume, surface and edge defects or any other defects):

$$\int_V \rho_{c,v}(V')\,dV' + \int_A \rho_{c,s}(A')\,dA' + \int_L \rho_{c,l}(L')\,dL' = \langle N_{c,v}\rangle + \langle N_{c,a}\rangle + \langle N_{c,l}\rangle = \langle N_{c,S}\rangle \quad (12)$$

Equation (11) is a very general strength distribution function which can be used to describe a wide variety of problems. The exponential law is a consequence of purely statistical arguments implicating no special size distribution of defects. Even for a high mean value, $\langle N_{c,S}\rangle \gg 1$, there is a (small) probability of finding specimens which do not contain any critical defect (e.g. for $\langle N_{c,S}\rangle = 5$, $R = \exp(-5) \simeq 0{\cdot}007$; one of 148 specimens will survive in this case). For a small mean value, $\langle N_{c,S}\rangle \ll 1$, the probability of failure is approximately equal to the mean number of critical defects per specimen, therefore it holds that

$$F_S = 1 - \exp(-\langle N_{c,S}\rangle) \simeq \langle N_{c,S}\rangle$$

(e.g. for $\langle N_{c,S}\rangle = 10^{-4}$, $F_S \simeq 10^{-4}$).

For a given stress amplitude the mean value defined in eqn (1) is monotonically increasing with the specimen size. The same is true for the probability of failure. For fracture mechanical reasons, the strength of a large defect is smaller than the strength of a small defect. Therefore the number of critical defects increases if the stress amplitude increases. The same holds for the mean number of critical defects and, using eqn (11), for the probability of failure. Therefore eqn (11) describes the two most significant features of the statistical nature of strength of brittle materials: the probability of failure increases with increasing specimen size and with increasing load amplitude.

The three assumptions made previously are necessary to link the probability of finding (not finding) a critical defect with the probability of failure (survival).

Assumption (a) is self-explanatory. In materials with a linear elastic behaviour (this is generally the case if brittle fracture occurs), it works well if the mean distance between defects is large compared to the diameter of the defects.

The weakest link hypothesis (assumption (b)) describes total brittle fracture of the material; it is a worst-case model. Breakdown in this assumption may arise if the elastic strain energy released by crack propagation starting from the defect is too low to cause the separation of the specimen into two or more pieces. This is favoured by several situations. Some cases have been observed in ceramics, where the fracture toughness increases with crack propagation due to interactions between crack planes (this is called R-curve behaviour; see Section 5) or where the fracture toughness varies spatially. Another example is loading with an inhomogeneous stress state, where the released strain energy decreases during crack propagation; this happens, for example, in the case of thermal shock loading.

Assumption (c) has been discussed at the beginning of this section. The density of critical defects is defined to be a mean taken over a large set of specimens. In all cases of technical importance, this density is very low in structural materials. Let it be assumed that 10^6 identical specimens with volume V are loaded with the same stress state and that 10^3 specimens fail. Then, roughly speaking, the mean value of critical defects within one specimen (volume V) loaded at this stress state is $10^3/10^6 = 10^{-3}$. The density of critical defects in a part of this specimen (volume) is $10^{-3}/\Delta V$.

3 Application to Weibull's Theory

3.1 Weibull statistics

Weibull evaluated fracture statistics for materials with a homogeneous (constant) defect density.[1,2] For a homogeneous uniaxial stress state, he showed that for materials containing volume defects the probability of failure is

$$F = 1 - \exp[-n(\sigma)V] \quad (13)$$

$n(\sigma)$ being a material function assumed to be independent from the position in the specimen and the direction of stress σ.[1] Weibull mentioned that $n(\sigma)$ can generally be any monotonically increasing function of stress. He showed that the function

$$n(\sigma) = \frac{1}{V_0}\left(\frac{\sigma - \sigma_u'}{\sigma_0'}\right)^{m'} \quad (14)$$

can be used to describe a wide variety of problems.[2] V_0 is an arbitrary normalizing volume (normally set to 1 mm^3), σ_u' is a lower bond of strength, and σ_0' and m' are material parameters. At $\sigma - \sigma_u' = \sigma_0'$ and for $V = V_0$, $F = 1 - \exp(-1) = 0{\cdot}63$.

The corresponding distribution function of strength is very flexible. It can be nicely fitted to many measured strength distributions of brittle materials. It can be shown, however, that using, for example, the Monte Carlo simulation technique described in Ref. 11, the lower bond of strength—especially in small data sets used in daily materials testing practice—is not a stable parameter. It depends to a large extent on the subset of specimens selected to measure the distribution function and not on its real value. Therefore, this lower bond is often set to zero ($\sigma_u' = 0$), giving the widely used two-parameter form of the Weibull distribution:

$$F(V, \sigma) = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad (15)$$

(If for a given set of tested specimens a three-parameter Weibull distribution function has been selected to fit the data in an optimal manner, the corresponding fit parameters σ_0' and m' are different from the parameters σ_0 and m of the best fitted two-parameter Weibull distribution. It holds generally that $m' \le m$.)

σ_0 is the characteristic strength of the material and m is a material parameter (called the Weibull parameter). These formulae can easily be deduced from eqn (11). This will be done in the following sections.

3.1.1 Homogeneous distribution of volume defects and homogeneous stress state

For a homogeneous distribution of volume defects and for a homogeneous stress state, the density of critical defects is also homogeneously distributed within the material. Therefore the expectation value

$$\langle N_{c,S} \rangle = \langle N_{c,V} \rangle = \rho_{c,V} V$$

is the density of critical volume defects. In general, this density depends on the applied stress amplitude. Inserting this expression in eqn (11) gives eqn (13), with $n(\sigma) = \rho_{c,V}$. To summarize, the following additional assumptions are necessary to obtain eqn (13):

(d) The defects are volume defects;
(e) they are homogeneously distributed within the material; and
(f) the stress state is homogeneous.

3.1.2 Homogeneous distribution of volume defects and inhomogeneous uniaxial stress state

For an inhomogeneous uniaxial stress state (e.g. in a bent beam), and using assumptions (a) to (f), the corresponding strength distribution is[1]

$$F(V, \sigma) = 1 - \exp\left[-\int_{V(\sigma>0)} n(\sigma)\,\mathrm{d}V\right] \quad (16)$$

The parameter σ designates a stress amplitude which characterizes the applied stress state. For example, in the case of a bent beam, σ can be defined to be the maximum outer fibre stress. The integration may only be made over regions with tensile stress components. This implies that

(g) compressive stresses are not damaging.

In most brittle materials, the compressive strength is more than five times higher than the tensile strength. Therefore assumption (g) is a fair approximation as long as the compressive stress amplitudes in a loaded structure are smaller or approximately equal to the tensile stress amplitudes.

3.1.3 Weibull statistics and defect size distribution

To work out an analytical form for the material function $n(\sigma)$, additional information concerning the shape, orientation and size distribution of the defects and on the microscopic fracture criterion has to be supplied. Again it is assumed for simplicity that the stress state is homogeneous and uniaxial.

To get the two-parameter Weibull distribution, eqn (15), it is additionally assumed that:

(h) The volume defects behave as flat cracks which can be characterized by a single variable (crack length a);
(i) they are oriented perpendicular to the applied stress direction;
(j) the frequency distribution density of defect lengths (mean number of defects per volume and defect length) is given by an inverse power law:

$$g(a) = Aa^{-r} \quad (17)$$

where A and r are material constants; and that
(k) the Griffith failure criterion applies:

$$K = \sigma Y \sqrt{a} \ge K_{Ic} \quad (18)$$

where K is the stress intensity factor, Y the geometry factor of the defect and K_{Ic} the fracture toughness.

(These assumptions are more restrictive than necessary; they are selected to make the calculation easy.)

Assumption (h) is only used to make the calculations easier, but it holds true in many cases. Important classes of flaws are spherical inclusions or pores remaining after the sintering. In the material around these flaws, there often exists a system of radial cracks. Then there can always be found a radial crack which is approximately perpendicular to any given stress direction, and assumptions (h) and (i) can be applied. Geometry factors for this crack configuration can be found in Refs 12 and 13.

For a given stress amplitude the critical defect size, a_c, depends, in general, on the nature of the defect, its shape and orientation, and on the fracture toughness of the material. Then more refined defect models have to be used which account for all these influences. In many cases, the proper choice of the defect model does not change the analytical structure of the Weibull distribution but the definition of the characteristic strength, σ_0.[14] This point will be briefly discussed in the next subsection.

The Griffith criterion (assumption (k); the sum of elastic energy released and the work done by crack propagation is higher than the energy needed to create new fracture surfaces) describes the onset of brittle fracture. It can easily be applied to brittle materials with a well-defined fracture toughness, e.g. to fine-grained non-reinforced ceramics. If the energy necessary to create new fracture surfaces increases with crack propagation (*R*-curve behaviour; this happens, for example, in transformation-toughened ceramics[15–18] or in coarse-grained ceramics[19–22]), the onset of brittle fracture need not cause catastrophic failure because stable crack growth may occur. In this case assumption (k) has to be replaced by a more refined failure criterion. This case will be discussed in Section 5.

With these assumptions ((h), (i), (k)) it can be shown that all cracks equal to or larger than a_c are critical:

$$a_c(\sigma) = \left(\frac{K_{Ic}}{\sigma Y}\right)^2 \tag{19}$$

To calculate the density of critical defects

$$n(\sigma) = \rho_{c,v}(\sigma) = \int_{a_c(\sigma)}^{\infty} g(a)\,\mathrm{d}a \tag{20}$$

the frequency distribution density of the defect length, $g(a)$, has to be known. This distribution function can either be measured (e.g. by metallographic (ceramographic) methods or using non-destructive testing procedures) or determined by

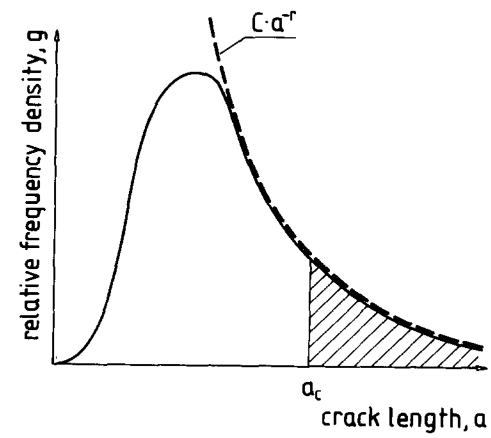

Fig. 1. Schematic drawing of the frequency distribution of the defect size; the right-hand side of the curve can be approximated by eqn (17). The shaded area is the density of critical defects (eqn (20)).

theoretical calculations. Measurements have only rarely been done. Poloniecki & Wilshaw measured this distribution curve of surface cracks in glass.[23] They determined the curve which is qualitatively shown in Fig. 1. It should be mentioned that the right-hand side of this curve can be described by an inverse power law. Sometimes only this side of the distribution curve can be observed. Wolf *et al.*[24] measured the size distribution of the radii of sintering pores in SiC in the radius interval between 2 and 200 μm. They obtained an inverse power law distribution function (corresponding to eqn (17)).[24] There also exist other cases where the frequency distribution density of the defect length has a different size dependency. If, for example, the defects are inclusions in interdentritic spaces, or if they are foreign powder particles found in a sieved powder compact, there must exist an upper limit for the size and, in this case, the density function can certainly not be approximated by an inverse power law.

Finally, it can be concluded that assumption (j) selects a special form of frequency distribution density of the defect length, which often, but not in all cases, can be observed in brittle materials.

Inserting eqn (17) into eqn (20), integrating and using eqn (19) gives

$$n(\sigma) = \rho_{c,v}(V,\sigma) = \frac{A}{r-1}(a_c)^{1-r} = \frac{1}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m \tag{21}$$

with

$$m = 2(r-1) \tag{22}$$

$$\sigma_0 = \frac{K_{Ic}}{Y}\left(\frac{r-1}{V_0 A}\right)^{1/m} \tag{23}$$

where V_0 is an arbitrary scaling parameter, often set to 1 mm^3. (It is assumed that the geometry factor, Y, is independent of crack length, a. This is approximately true for small cracks ($a/W \ll 1$; W: typical specimen dimension). This is generally the case in brittle materials.)

Inserting eqn (21) into eqn (13) gives the two-parameter Weibull distribution, eqn (15). As also stated in earlier papers,[6,25] the Weibull modulus m only depends on the material parameter r (eqn (22)), which describes the size dependency of the relative frequency of the crack lengths.

For an inhomogeneous uniaxial stress state, eqn (15) can be transformed into an integral. This gives eqn (16), with $n(\sigma) = (1/V_0)(\sigma/\sigma_0)^m$.

3.1.4 Cracks with generally distributed crack plane orientations; uniaxial and general stress state

For a crack of given size, the crack orientation has a large influence on the risk of failure. In a uniaxial stress state, a crack which is perpendicular to the stress direction may be critical, depending on the local stress amplitude and on crack size. However, if the crack of the same size loaded with the same stress amplitude lies parallel to the stress direction, it is harmless, because its stress intensity factor is zero (if only mode I loading is concerned). For generally orientated cracks two problems have to be handled.

Firstly, normal and shear stresses are acting in the crack plane, causing modes I, II and III loading of the crack. Therefore a mixed mode microscopic fracture criterion has to be worked out, even if the macroscopic stress state is uniaxial. Using this criterion an equivalent stress σ_e can be defined, which depends on the normal and on the shear stress components. Then the stress, σ, in the Weibull distribution, eqn (15), can be replaced by the equivalent stress, σ_e. In the context of a fracture mechanical concept of mixed mode loading,[26,27] several different failure criteria have been proposed.[28-31] None of them is shown to describe the material's behaviour in full detail. Therefore, this problem is still a subject of intensive research. A good overview of this problem is given by Thiemeier in his thesis.[14]

Secondly, the critical crack length depends not only on the equivalent stress amplitude (and therefore on the fracture criterion) but also on the orientation of the crack plane. Therefore not all cracks (as in the example shown in Section 3.1.3) but only a fraction of the cracks of a given size can be critical. For each equivalent stress amplitude and each crack length, a region of space angles can be found, in which the cracks are critical. If the microscopic failure criterion is selected, this problem can easily be solved. An example is given in Ref. 6. If the other conditions mentioned before are not changed, the corresponding fracture statistics are again Weibull-type statistics (analogous to eqns (15) and (16)). The significance of the Weibull modulus does not change but the parameter σ_0 has to be redefined in order to account for the influence of the crack plane orientations and of the mixed mode loading of the cracks.

3.1.5 Conditions for obtaining a Weibull-type strength distribution function

The strength of brittle materials is always of the Weibull type (eqn (15)), if the defects are sparsely and homogeneously distributed in the volume, and if the density of critical defects, $\rho_{c,v}$, is proportional to some power of the stress amplitude. This density depends on the frequency distribution density of defect lengths and on the stress dependence of the critical defect length (see eqn (21)). The latter has to be some power of the stress amplitude, which approximately holds true in most cases of brittle fracture. For example, if the Griffith fracture criterion applies, $a_c \propto \sigma^{-2}$ (eqn (19)). Therefore, the most important assumption which has to be satisfied in order to obtain a Weibull distribution of strength is that the frequency distribution density of defect length is an inverse power function of the defect length:

$$g(a) = Aa^{-r}$$

This often (at least in a limited interval of crack lengths) but not in all cases occurs in brittle materials. Therefore the Weibull distribution is a special case of a more general distribution function (eqn (8)) for a special type of frequency distribution of the defect length.

The question remains why Weibull distributions are measured so often in daily materials testing practice. In this practice the set of specimens used to measure the strength distribution is in general very small. The number of test specimens fractured is between 10 and 30 in most cases; in rare cases it is up to 100. Let us consider the example of a data set with 100 specimens and a material with a typical Weibull modulus of $m = 10$. Then the probability of failure (approximately given by $(n - 0{\cdot}5)/N$, where n is the ranking number of the test specimen) of the weakest specimen is 0·995, and that of the strongest specimen is 0·005. Using eqn (15) and for tensile specimens with the volume $V = V_0$, the strength of the strongest specimen is about twice the strength of the weakest specimen. (From

$$F = 1 - \exp\left[-(V/V_0)(\sigma/\sigma_0)^m\right]$$

Table 1. Size ratio of the weakest and the strongest defect of a material with Weibull modulus m in a set of data containing N specimens

m	$a_{c,w}/a_{c,s}$		
	N		
	10	*30*	*100*
10	2·3	3·0	4·0
15	1·7	2·1	2·5
20	1·5	1·7	2·0

it follows that at $V = V_0$

$$(1/m) \ln \ln [1/(1-F)] = \ln(\sigma/\sigma_0)$$

This formula can be evaluated for the weakest (w) and for the strongest (s) specimen of the data set. The difference of both equations gives

$$(1/m)\{\ln \ln [1/(1-F_w)] - \ln \ln [1/(1-F_s)]\} = \ln(\sigma_w/\sigma_s).)$$

Using eqn (19), the size of the critical defect in the weakest specimen is about four times the size of the critical defect in the strongest specimen. Table 1 lists the size ratios for data sets containing a different number of test specimens, N, and materials with two different Weibull moduli ($m = 10$ is typical for many ceramic materials, $m = 20$ corresponds to high-quality ceramics). For small data sets, this ratio is always small. In bending specimens made from ceramic materials, for example, the size of a typical fracture origin is 50 μm and the sizes of the smallest and biggest fracture origins are about 30 and 80 μm respectively. So the size interval of the critical defects is relatively small. A Weibull distribution function is observed if in this size interval the density of critical defects follows a power law of a. Due to the inherent scatter of the data it is always possible to fit a power law function to a small set of data points within a small size interval, and therefore a Weibull distribution of strength is so often observed.

3.1.6 Limitations of data extrapolation

In general, the volume of test specimens is different from that of components, and the same holds for the stress state. Applying Weibull statistics for components, it is helpful to introduce the 'effective volume' in order to take these differences into account. The effective volume, V_{eff}, of a test specimen (component) is the volume corresponding to the gauge section of a hypothetical tensile specimen loaded at the maximum equivalent stress amplitude, $\sigma_{e,max}$, occurring in the component. Its size is selected in such a way that the reliability of the hypothetical tensile specimen and of the component are equal:

$$V_{eff} = \left(\frac{\sigma_0}{\sigma_{e,max}}\right)^m \int_V \left(\frac{\sigma_e}{\sigma_0}\right)^m dV \tag{24}$$

It depends on the Weibull modulus and on the stress state. If large stress gradients occur, the effective volume can be much smaller than the real volume. In the case of three-point bend testing of typical ceramics, $V_{eff}/V \approx 1/1000$ and, for four-point bend testing, $V_{eff}/V \approx 1/100$.[25,32] In the case of tensile testing, however, V_{eff}/V can be 1/3 and even more. (Bars with rectangular cross-section (as commonly used for bend testing) were glued into steel fixtures and tensile tested. The material necessary for gripping was minimized and $V_{eff}/V > 1/3$ (Danzer, R., unpublished).)

The following arguments to discuss the limitations of data extrapolation are guided by the idea that the size of the critical defect in the weakest specimen (it is the largest defect) should be equal to the size of the critical defect in the weakest component. Then the range of experience (concerning the size distribution of defects and strength distribution of specimens) gained from the set of test specimens embraces the data necessary to predict the component behaviour.

When the Weibull statistics are used to predict the reliability of components, a high component reliability, R_{comp} (e.g. $R_{comp} = 0{\cdot}999\,99$, $0{\cdot}999\,999$ or even higher; the corresponding probability of failure is $F_{comp} = 10^{-5}$, 10^{-6} or lower), is required in most cases. Using Weibull's theory, this can only be done in a reliable way if the fracture origins in the test specimens and components belong to the same flaw population, and if their size distribution follows a power law (eqn (17)). Of course, the defect density distribution has to be equal in specimens and components. This can be guaranteed in the best way if the specimens are cut out of the components.

Let us at first consider the problem when the effective volume of the specimens equals that of the components. If the set of test specimens used to measure the Weibull distribution contains N specimens, the range of 'measured' reliabilities (as mentioned earlier, an estimate for the specimen's reliability is $R = 1-(n-0{\cdot}5)/N$, where n is the ranking number of test specimen) goes from $(2N-1)/2N$ (for $n = 1$) to $1/2N$ (for $n = N$). The latter value corresponds to the specimen containing the largest defect. In the case of equal effective volumes, the size of the largest defects in both sets is equal, if the number of test specimens is $N_{min} = 1/(2F_{comp}) = 1/(1-R_{comp})$. In this case, $N = N_{min}$ is the minimum number of test specimens used to 'measure' the required reliability R_{comp}. For example, for $R_{comp} = 0{\cdot}999\,99$ ($F_{comp} = 10^{-5}$), the minimum number of specimens to measure this reliability value is 5×10^4.

So far no extrapolation has been made, but it is possible to use specimens with a geometry and a loading state different to those of the components as long as the effective volume remains approximately the same. The calculation is based on the two-parameter Weibull distribution. Therefore, the argument remains true as long as the set of $N = 1/(2F_{comp})$ specimens is well described by the two-parameter Weibull distribution.

To account for the different effective volumes of specimens and components, the number of necessary test specimens has to be multiplied by the ratio of the effective volumes of the components and specimens.[33] Then the number of necessary specimens is

$$N_{min} = (V_{eff,comp}/V_{eff,spec})/(2F_{comp})$$

If large stress gradients occur, which is the case in many components, the effective volume is much smaller than the real volume. Then, for example, using the tensile testing technique mentioned previously, the effective volume of the specimens may be equal to or even much greater than that of the components. This may also happen if specimens were cut out of the components. Continuing the above example it is assumed that $V_{eff,spec}/V_{eff,comp} = 10$. This reduces the number of necessary specimens to 5000.

Until now no real extrapolations have been performed, because the description of both the specimens and the components is based on the knowledge of the defect size distribution within the same size interval. If it is assumed that the size distribution does not change spontaneously beyond the largest measured defect size, the data can be extrapolated and the number of necessary specimens can be further reduced, say, by an empirically selected factor of $1/\alpha$. The parameter α is the extrapolation span. Table 2 lists the ratio of the largest defect corresponding to the required components' reliability ($a_{c,w,comp}$) and of the largest defect ($a_{c,w,spec}$) expected in a set of test specimens, depending on the extrapolation span and on the Weibull modulus, m. The calculation is based on the two-parameter Weibull distribution.

Table 2. Size ratio of the weakest defect, $a_{c,w,comp}$, corresponding to the required component reliability and of the weakest defect, $a_{c,w,spec}$, in a set of test specimens

	$a_{c,w,comp}/a_{c,w,spec}$				
	α				
m	3	5	10	20	50
10	1·25	1·38	1·58	1·82	2·19
15	1·16	1·24	1·36	1·49	1·68
20	1·12	1·17	1·26	1·35	1·49

If it is assumed that the expected defect size in the components should not be much larger than the observed defect size in test specimens, this table is a basis for the definition of the tolerable extrapolation span. In the author's opinion it should be tolerable to have components containing defects up to 50% larger than those observed in test specimens, giving $a_{c,w,comp}/a_{c,w,spec} = 1{\cdot}5$. In the framework of the two-parameter Weibull theory these assumptions yield, for the maximal tolerable extrapolation span, $\alpha_{max} \simeq 1{\cdot}5^{m/2}$.

Use of these assumptions gives

$$N_{min} = \frac{V_{eff,comp}}{V_{eff,spec}} \frac{1}{2\alpha F_{comp}} \approx \frac{1}{2(1{\cdot}5)^{m/2}} \frac{V_{eff,comp}}{V_{eff,spec}} \frac{1}{(1 - R_{comp})} \qquad (25)$$

In the above example, and for $m = 15$ ($\alpha = 20$), the number of necessary test specimens is reduced to about 250.

In line with these arguments, the Weibull statistics can be used to calculate the component reliability if, within a set of N_{min} test specimens, no indication of any deviation from Weibull's theory can be found.

4 Volume and Surface Defects

In brittle materials surface cracks are often produced during machining of the specimens. The two (or more) different defect populations (volume defects and surface defects) occur. The corresponding probability of failure (see eqns (9) and (12)) is

$$F_S = 1 - \exp\left[-\langle N_{c,v}\rangle - \langle N_{c,s}\rangle\right] \qquad (26)$$

The subscripts v and s refer to the volume and the surface respectively. If, for each of both defect populations, the conditions for obtaining a Weibull distribution (see Section 3.1.5) are valid, this equation can be worked out to give[25]

$$F_S = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_{0,v}}\right)^{m_v} - \frac{S}{S_0}\left(\frac{\sigma}{\sigma_{0,s}}\right)^{m_s}\right] \qquad (27)$$

where S is the surface under load, S_0 is a normalizing surface normally set to $1\,mm^3$, and $\sigma_{0,s}$ is the characteristic strength for a material containing only surface defects. This type of strength distribution has recently been observed in zirconia ceramics.[34] If the surface finish is insufficient (this is often the case with brittle materials), $\langle N_{c,s} \gg \langle N_{c,v} \rangle$, the strength is dominated by surface cracks.

5 Increase of Fracture Toughness during Crack Extension

A fracture toughness which increases as the crack grows has been observed in several materials, e.g. in transformation-toughened zirconia[15–18] and in coarse-grained alumina.[19–22] This increase is caused by the interaction of the crack surfaces and depends, therefore, on the actual crack and specimen geometry. The crack resistance, $\tilde{R}$ ($\tilde{R}$: energy absorbed by the newly created cracked area), is strongly related to the fracture toughness, K_{Ic}:

$$E'\tilde{R} = K_{Ic}^2$$

$E' = E$ in a plane stress state and $E' = E/(1-\nu^2)$ in a plane strain state, where E is the elastic modulus and ν is the Poisson ratio. A typical R-curve is shown in Fig. 2, where the crack resistance, R, is plotted over the crack advance, Δa (solid line). In the linear elastic limit, the energy released by the crack advance, G, is

$$G = K^2/E' = \sigma^2 Y^2 a/E'$$

For small cracks, Y is independent of the crack length, giving a linear relationship between the released energy and the crack length. For a crack of length $a_i + \Delta a$ (a_i is the crack length before crack advance occurs), this is shown in Fig. 3 (dashed lines). The slope of the lines ($\sigma^2 Y^2/E'$) depends on the applied stress amplitude, σ. For a small stress amplitude, the released energy is too small to balance the energy necessary to create the crack advance (line 1). If the stress is increased (line 2), the crack may grow in a stable manner but it stops when the toughness increases and $\tilde{R}$ exceeds G. Unstable crack growth is only possible if the stress is increased further to the state shown in line 3, where, for each crack advance, the released energy is greater than or equal to the absorbed energy. This gives, in addition to eqn (18) ($G > \tilde{R}$), the condition

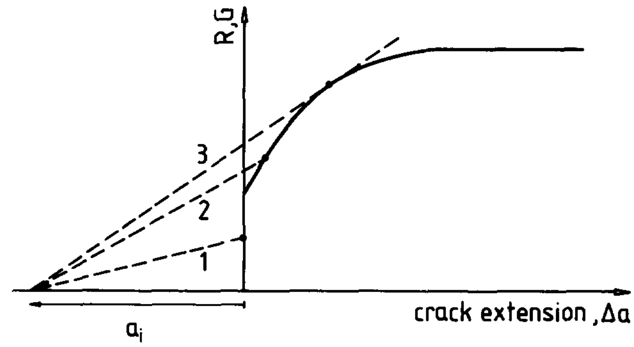

Fig. 3. Crack advance occurs if the released strain energy, G (dashed lines; the ordering number indicates an increasing stress amplitude), exceeds the crack resistance, $\tilde{R}$.

$$\frac{dG}{da} \geq \frac{d\tilde{R}}{da} \tag{28}$$

R_0 is defined as (see Fig. 2) the minimum crack resistance (at $\Delta a = 0$) and $R'(0)$ is its slope at $\Delta a = 0$. The length $l = R_0/R'$ is a measure of the size of the interaction zone. The initial crack sizes (before stable crack growth occurs) can be divided into three classes:

$$a_i = \begin{cases} \leq l \\ \text{between } l \text{ and } n_0 l \\ > n_0 l \end{cases} \tag{29}$$

n_0 is a number of the order of 10. In the first case, the initial cracks are so small that the stresses necessary to initiate crack propagation are very high. Stable crack growth is not possible and running cracks do not arrest. The crack resistance is $\tilde{R} = R_0$ and the ideas of Section 3.1 remain valid. This will occur in many transformation-toughened ceramics, where the interaction zone can reach several millimetres and the size of natural flaws is one or two orders of magnitude smaller.

For cracks in the second size interval, the crack resistance as well as the crack size increase before catastrophic failure occurs. The critical crack size does not depend on some power of the applied stress (a more complex relationship exists), and the corresponding probability of failure is not a Weibull distribution (eqn (15)). In general, the increase of crack resistance causes a systematic decrease of the scatter of the strength data. In the extreme, the strength can approximately be independent of the size of the fracture initiating defects.

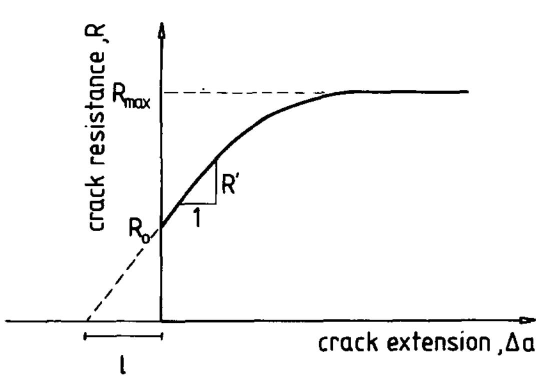

Fig. 2. Schematic sketch of an $\tilde{R}$-curve (energy absorbed per newly fractured area over crack extension) for a coarse-grained ceramic material. The length $l = R_0/R'$ is a measure of the length of the interaction zone.

In the third size interval, the cracks are much larger than the size of the interaction zone. A very small crack growth can cause an increase in the crack resistance to its upper plateau value, R_{max}. In a good approximation, the material behaves as a material with a flat crack resistance curve and $\tilde{R} = R_{max}$. In this case, the arguments of Section 3.1 remain valid as well.

Summarizing the above arguments, it can be said that for materials with a rising crack resistance curve, a Weibull distribution can only be expected in distinct stress intervals (corresponding to the crack size intervals), and extrapolations based on the Weibull distribution may give wrong results. More general statistics of strength based on the failure criterion already discussed (eqn (28)) and on the R-curve have to be used.

6 Final Remarks

The new fracture statistics, eqn (11), form a very simple and general function, which applies to many situations occurring in brittle materials. It correlates the probability of failure with the expectation value of finding critical defects in a specimen (component). Because of its simplicity, it opens a simple way to win insight into the fracture statistical problems. To apply these statistics to material problems the expectation value of finding critical defects has to be evaluated in terms of the applied stress state. Doing this, micromechanical models concerning the fracture criterion and depending on the kind of flaw, its size, size distribution and on the flaw orientation have to be formulated. This opens a wide field for future research.

The well-known Weibull statistics are a special case of the new statistics and, therefore, need not be applied to each material. From experience it is known that many sets of test data can be described by Weibull statistics. This happens due to their flexibility in data fitting, the inherent scatter of data and the small size (interpolation range) of most data sets. Weibull distributions measured in such a way can be used for data interpolations but should only be used with care for data extrapolations. Doing this, the minimum number of test specimens necessary to guarantee a component reliability can be defined, depending on the ratio of the effective volumes of components and test specimens on the one hand and on the required reliability of the component on the other. A large effective volume of the test specimens reduces the necessary number of test specimens. This is a strong argument for the introduction of tensile testing in the testing practice of ceramics. Such deviations are expected to occur, for example, in materials where critical surface and volume defects are active or in materials with a rising crack resistance curve.

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